A Theoretical Analysis of Catastrophic Forgetting

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Doan, T., Bennani, M. A., Mazoure, B., Rabusseau, G. & Alquier, P. (2021). A theoretical analysis of catastrophic forgetting through the NTK overlap matrix. *AISTATS'2021*.

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Notations

Regression/classification problem :

- objects $x \in \mathcal{X}$,
- labels $y \in \mathcal{Y} \subset \mathbb{R}$,
- predictors $f_w : \mathcal{X} \to \mathcal{Y}$, $w \in \mathbb{R}^d$, objective : neural networks.

Difficulties of "continual learning"

- d is huge, $\to$ we need a lot of data.
- the dataset is huge, $\to$ impossible to store all the data.
- we will learn w sequentially based on a data stream (x_t, y_t), $\to$ the x_t come from a real life data collection process that makes them non-indentically distributed..

Online learning theory

Online learning theory provides algorithms to learn from data streams, with theoretical guarantees.

Online Gradient Algorithm

- $w_1 := 0$,
- $w_{t+1} = w_t - \eta_t \nabla_{w=w_t} \ell(y_t, f_w(x_t))$.

Regret bound for OGA

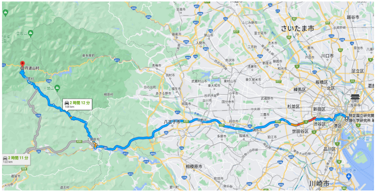
If ℓ is L-Lipschitz + convex, one can calibrate η_t such that

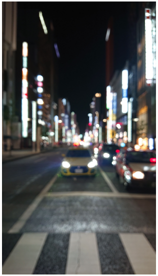
$$\frac{1}{T}\sum_{t=1}^{T} \ell(y_t, f_{w_t}(x_t)) - \inf_{\|w\|\leq B} \frac{1}{T}\sum_{t=1}^{T} \ell(y_t, f_w(x_t)) \leq BL\sqrt{\frac{2}{T}}.$$

Example : training a self-driving car

Decide an itinerary

- from RIKEN AIP (Tokyo)
- to Tabayama.





Observation

$$y_t = f_{w^*}(x_t)$$

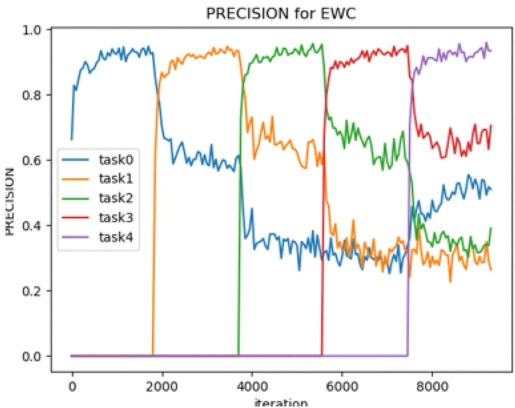
- $y = 1, \dots, \tau_1$:
  - x_t i.i.d from $P_1 \rightarrow$ we learn w_1.
- $y = \tau_1 + 1, \dots, \tau_2$:
  - x_t i.i.d from $P_2 \rightarrow$ we update w_1 to w_2.
- $\dots$
- $y = \tau_K + 1, \dots, \tau_{K+1}$:
  - x_t i.i.d from $P_K \rightarrow$ we update w_K to w_{K+1}.
- $x \sim P_1$:
  - $f_{w_{K+1}}(x)$ is a much worse prediction than $f_{w_1}(x)$.
  - we forgot how to deal with objects $x \sim P_1$.

What is the problem with online learning theory ?

$$\frac{1}{T}\sum_{t=1}^{T}\ell(y_t, f_{w_t}(x_t)) - \inf_{\|w\|\leq B}\frac{1}{T}\sum_{t=1}^{T}\ell(y_t, f_w(x_t)) \leq BL\sqrt{\frac{2}{T}}.$$

- tells you $f_{w_t}(x_t)$ predicts well y_t (on average over t), *not* that $f_{w_T}(x_t)$ predicts well y_t.
- *online-to-batch* bounds : averaging $\bar{w}_t = \frac{1}{t}\sum_{s=1}^{t} w_s$ is proven to work well for out-of-sample prediction... in the i.i.d case !

An example



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Some references

Sutton, R. (1986). Two problems with back propagation and other steepest descent learning procedures for networks. *Proceedings of the Eighth Annual Conference of the Cognitive Science Society*.

French, R. M. (1999). Catastrophic forgetting in connectionist networks. *Trends in cognitive sciences*.

Kirkpatrick, J., Pascanu, R., Rabinowitz, N., Veness, J., Desjardins, G., Rusu, A. A., Milan, K., Quan, J., Ramalho, T., Grabska-Barwinska, A. & Hassabis, D. (2017). Overcoming catastrophic forgetting in neural networks. *Proceedings of the National Academy of Sciences*.

Kemker, R., McClure, M., Abitino, A., Hayes, T. & Kanan, C. (2018). Measuring catastrophic forgetting in neural networks. *AAAI'2018*.

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Linear model – notations

- initialization : $w_{\tau_0} = 0$.
- task τ_k given as a block :

$$\mathrm{Y}_{\tau_k} := \begin{pmatrix} y_{\tau_k+1} \\ \vdots \\ y_{\tau_{k+1}} \end{pmatrix} \text{ and } \mathrm{X}_{\tau_k} := \begin{pmatrix} x_{\tau_k+1}^T \\ \hline \vdots \\ \hline x_{\tau_{k+1}}^T \end{pmatrix}$$

- update :

$$\begin{aligned} w_{\tau_k} &= \arg\min_{w \in \mathbb{R}^d} \left\{ \|\mathrm{Y}_{\tau_k} - \mathrm{X}_{\tau_k} w\|^2 + \lambda . \|w - w_{\tau_{k-1}}\|^2 \right\} \\ &= w_{\tau_{k-1}} + (\mathrm{X}_{\tau_k}^T \mathrm{X}_{\tau_k} + \lambda . \mathrm{I})^{-} \mathrm{X}_{\tau_k}^T \underbrace{\left(\mathrm{Y}_{\tau_k} - \mathrm{X}_{\tau_k} w_{\tau_{k-1}} \right)}_{= \tilde{\mathrm{Y}}_{\tau_k}} . \end{aligned}$$

Definition of forgetting

Definition - forgetting of task *i* at the end of task *j*

For $s \leq t$ we put

$$\Delta^{\tau_s \to \tau_t} := \|X_{\tau_s} w_{\tau_t} - X_{\tau_s} w_{\tau_s}\|^2.$$

- $X_{\tau_t} = U_{\tau_t} \Sigma_{\tau_t} V_{\tau_t}^T$ be the SVD of X_{τ_t},
- $O^{\tau_s \to \tau_t} = V_{\tau_s}^T V_{\tau_t}$ the overlap matrix,
- $M_{\tau_t} := \Sigma_{\tau_t} (\Sigma_{\tau_t} + \lambda.I)^{-} U_{\tau_t}^T$.

Theorem

For any $t > s$,

$$\Delta^{\tau_s \to \tau_t} = \left\| \sum_{k=s+1}^{t} U_{\tau_k} \Sigma_{\tau_k} O^{\tau_s \to \tau_k} M_{\tau_k} \tilde{Y}_{\tau_k} \right\|^2 .$$

Upper bound on forgetting

Corollary

$$\sqrt{\Delta^{\tau_s \to \tau_t}} \leq \|\Sigma_{\tau_s}\|_{\mathrm{op}} \sum_{k=s+1}^{t} \|O^{\tau_s \to \tau_t}\|_{\mathrm{op}} \left\| M_{\tau_k} \tilde{Y}_{\tau_k} \right\|$$

With $V_{\tau_t} = (V_{\tau_t}[1] | V_{\tau_t}[2] | \dots)$ we have

$$O_{i,j}^{\tau_s \to \tau_t} = \cos(V_{\tau_s}[i], V_{\tau_t}[j])$$

and $\|O^{\tau_s \to \tau_t}\|_{\mathrm{op}} = \cos(\alpha)$ where α is the Dixmier angle between the span of V_{τ_t} and the span of V_{τ_s}.

Dixmier, J. (1949). Étude sur les variétés et les opérateurs de Julia, avec quelques applications. *Bulletin de la SMF*.

A recent improvement

Evron, I., Moroshko, E., Ward, R., Srebro, N. & Soudry, D. (2022). How catastrophic can catastrophic forgetting be in linear regression ? *COLT'22*.

- simplified setting, allows an refinement of the analysis,
- note : I find their results very elegant, so I presented the previous result using *some* of their notations.

In their paper :

- $\lambda = 0$, there is w^* such that $\mathrm{Y}_{\tau_s} = \mathrm{X}_{\tau_s} w^*$ (no noise).
- the X_{τ_s} are normalized $\Rightarrow \|\Sigma_{\tau_s}\|_{\mathrm{op}} \leq 1$.

Consequences of the simplifications

Define the orthogonal projection $\mathrm{P}_{\tau_k} = I - \mathrm{X}_{\tau_k}(\mathrm{X}_{\tau_k}^T \mathrm{X}_{\tau_k})^{-} \mathrm{X}_{\tau_k}^T$,

$$\text{then } w_{\tau_k} - w^* = \mathrm{P}_{\tau_k}(w_{\tau_{k-1}} - w^*) = \mathrm{P}_{\tau_k} \dots \mathrm{P}_{\tau_1}(\underbrace{w_{\tau_0}}_{=0} - w^*),$$

$$\begin{aligned} \text{and } \Delta^{\tau_s \to \tau_t} &= \|\mathrm{X}_{\tau_s} w_{\tau_t} - \mathrm{X}_{\tau_s} w_{\tau_s}\|^2 \\ &= \|\mathrm{X}_{\tau_s} w_{\tau_t} - \mathrm{Y}_{\tau_s}\|^2 \\ &= \|\mathrm{X}_{\tau_s} w_{\tau_t} - \mathrm{X}_{\tau_s} w^*\|^2 \\ &= \|\mathrm{X}_{\tau_s} \mathrm{P}_{\tau_t} \dots \mathrm{P}_{\tau_1} w^*\|^2 \\ &\leq \|(\mathrm{I} - \mathrm{P}_{\tau_s}) \mathrm{P}_{\tau_t} \dots \mathrm{P}_{\tau_1} w^*\|^2. \end{aligned}$$

Average forgetting : worst case

Definition - average forgetting at task t

$$F(t) := \frac{1}{t}\sum_{s=1}^{t} \|X_{\tau_s} w_{\tau_t} - X_{\tau_s} w_{\tau_s}\|^2 = \frac{1}{t}\sum_{s=1}^{t} \Delta^{\tau_s \to \tau_t}$$

$$F(t) = \frac{1}{t}\sum_{s=1}^{t} \|X_{\tau_s} P_{\tau_t} \dots P_{\tau_1} w^{*}\|^2$$

They design a situation where :

$$F(t) \geq 1 - \mathcal{O}\left(\frac{1}{\sqrt{t}}\right).$$

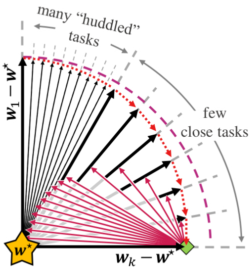

Figure from Evron *et al.* (2022).

Situations where forgetting do not occur

Evron *et al.* (2022) then argue that in general, forgetting is not that bad :

- cyclic tasks : $\tau_1, \dots, \tau_T, \tau_1, \dots, \tau_T, \dots$ After seeing t tasks,

$$F(t) \leq \min\left(\frac{T^2}{\sqrt{t}}, \frac{T^2(d - \max\{\mathrm{rank}(X_{\tau_s})\}}{t} \right),$$

- randomized tasks : $\tau_{I_1}, \tau_{I_2}, \dots$ where the I_i are i.i.d uniform in $\{1, \dots, T\}$, then after seeing t tasks,

$$\mathbb{E}[F(t)] \leq \frac{9\left(d - \frac{1}{T}\sum_{s=1}^{T} \mathrm{rank}(X_{\tau_s})\right)}{t}.$$

→ however, this requires to store the tasks, or, at least, to be able to learn them many times...

Conclusion of the theoretical analysis

What we learnt so far

- catastrophic forgetting can happen even in linear models,
- depends on the geometry and order of the tasks.

Open questions :

- noisy case,
- nonlinear case,
- tasks not by block // not aware that a new task begins,
- other algorithms... (we propose a few in the next section),
- theoretical limitations :

Knoblauch, J., Hisham, H. & Diethe, T. (2020). Optimal continual learning has perfect memory and is NP-hard. *ICML'2020*.

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Orthogonal updates

Doan, T., Bennani, M. A. & Sugiyama, M. (2020). Generalisation guarantees for continual learning with orthogonal gradient descent. *ICML'2020 Workshop on Lifelong Learning*.

$$w_{\tau_k} = \underset{\substack{w \in \mathbb{R}^d \\ V_{\tau_1}^T (w - w_{\tau_{k-1}}) = 0 \\ \vdots \\ V_{\tau_{k-1}}^T (w - w_{\tau_{k-1}}) = 0}}{\arg\min} \left\{ \|Y_{\tau_k} - X_{\tau_k} w\|^2 + \lambda . \|w - w_{\tau_{k-1}}\|^2 \right\}$$

$$= w_{\tau_{k-1}} + \Pi_k (X_{\tau_k}^T X_{\tau_k} + \lambda . I)^{-} X_{\tau_k}^T \left(Y_{\tau_k} - X_{\tau_k} w_{\tau_{k-1}} \right)$$

where Π_k is the orthogonal projection on $\ker(V_{\tau_1}^T | \dots | V_{\tau_{k-1}}^T)$.

$$\Delta^{\tau_s \to \tau_t} = 0.$$

But the procedure requires to store V_{τ_1}, V_{τ_2}, ...

Data compression (1/2)

In general, $\underbrace{\mathrm{X}_{\tau_t}}_{N_t \times d} = \underbrace{\mathrm{U}_{\tau_t}}_{N_t \times N_t} \underbrace{\Sigma_{\tau_t}}_{N_t \times N_t} \underbrace{\mathrm{V}_{\tau_t}^T}_{N_t \times d}$.

Data compression : replace V_{τ_t} by $\hat{\mathrm{V}}_{\tau_t}$ $(d \times n,\ n \ll N_t)$:

- "OGD" : $\hat{\mathrm{X}}_{\tau_t}$: n rows sampled from X_{τ_t}, $\hat{\mathrm{X}}_{\tau_t} = \hat{\mathrm{U}}_{\tau_t} \hat{\Sigma}_{\tau_t} \hat{\mathrm{V}}_{\tau_t}^T$.

Farajtabar, M., Azizan, N., Mott, A. & Li, A. (2020). Orthogonal gradient descent for continual learning. *AISTATS'2020*.

- instead of random rows, "memorable observations" :

Pan, P. , Swaroop, S. , Immer, A., Eschenhagen, R., Turner, R. & Khan, M. E. (2020). Continual Deep Learning by Functional Regularisation of Memorable Past. *NeurIPS'2020*.

Different framework, but the philosophy would here lead to select high-leverage observations.

Data compression (2/2)

Data compression : replace V_{τ_t} by $\hat{V}_{\tau_t}$ $(n \times d,\ n \ll N_t)$:

- our proposal, "PCA-OGD" : PCA on X_{τ_t}, that is

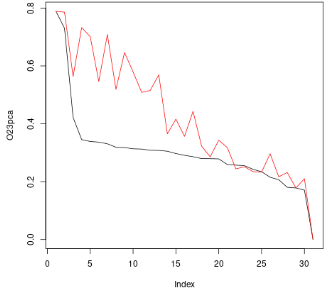
$$V_{\tau_t}^T = \left(\frac{\hat{V}_{\tau_t}^T}{*} \right).$$

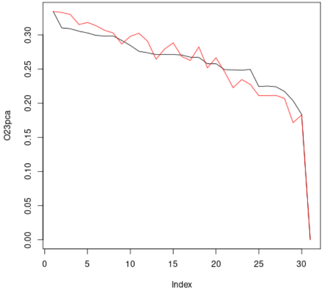
- $\hat{\Pi}_t :=$ orthogonal projection on $\ker(\hat{V}_{\tau_1}^T | \dots | \hat{V}_{\tau_{t-1}}^T)$.
- $\hat{O}^{\tau_s \to \tau_t} = V_{\tau_s}^T \hat{\Pi}_t V_{\tau_t}$

$$\sqrt{\Delta^{\tau_s \to \tau_t}} \leq \|\Sigma_{\tau_s}\|_{\mathrm{op}} \sum_{k=s+1}^{t} \left\| \hat{O}^{\tau_s \to \tau_t} \right\|_{\mathrm{op}} \left\| M_{\tau_k} \tilde{Y}_{\tau_k} \right\|$$

Simulation

$\|\hat{O}^{\tau_s \to \tau_t}\|_{\mathrm{op}}$ for "OGD" and "PCA-OGD" in two settings.





Experiments on the MNIST dataset

Neural network with the NTK approximation :

$$f_w(x) \simeq f_{w_0}(x) + \langle \nabla_{w=w_0} f_w(x), w - w_0 \rangle$$

Experiments : impact of $\|O^{\tau_s \to \tau_t}\|_{\mathrm{op}}$

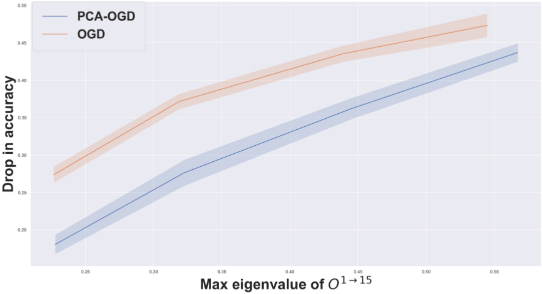
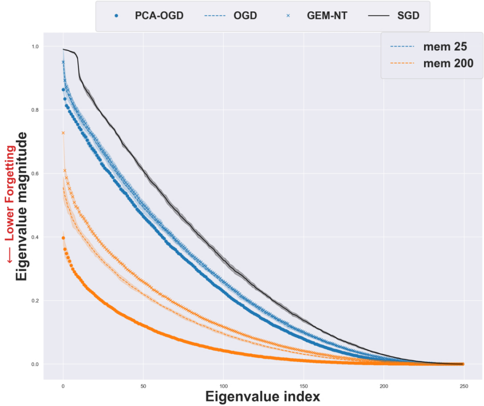


Figure 2: Drop in performance with respect to the maximum eigenvalue for Rotated MNIST (averaged over 5 seeds ± 1 std).

Experiments : evaluation of $\|\hat{O}^{\tau_s \to \tau_t}\|_{op}$



Experiments : performances

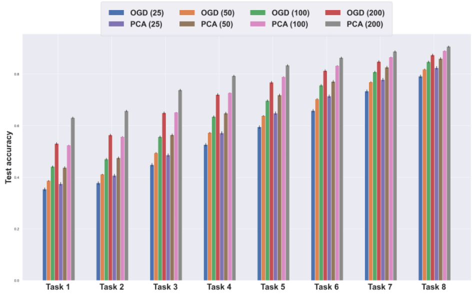


Figure 4: Final accuracy on **Rotated MNIST** for different memory size (averaged over 5 seeds ± 1 std). OGD needs twice as much memory as PCA-OGD in order to achieve the same performance (i.e compare OGD (200) and PCA (100).

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終わり

ありがとう ございます。